Chapter 14 - Bessel Functions

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Exercises: 14.1

`14.1.3`

Using only the generating function

$$e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n \tag{1}$$

and not the explicit series form of $J_n(t)$, show that $J_n(t)$ has odd and even parity according to whether *n* is odd or even, that is,

$$J_n(x) = (-1)^n J_n(-x) \tag{2}$$

Solution:

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} \tag{3}$$

$$\begin{aligned}
J_n(x) &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} && (4)\\
J_n(-x) &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{-x}{2}\right)^{n+2s} && (5)\\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(-1)^{n+2s}\left(\frac{x}{2}\right)^{n+2s} && (6)\\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(-1)^n(-1)^{2s}\left(\frac{x}{2}\right)^{n+2s} && (7)\\
&= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} && (8)\\
J_n(-x) &= (-1)^n J_n(x) && (9)\\
\Rightarrow (-1)^n J_n(-x) &= J_n(x) && (10)
\end{aligned}$$

`14.1.4`

Use the basic recurrence formulas,

$$\begin{aligned} J_{n-1}(x) + J_{n+1}(x) &= \frac{2n}{x} J_n(x) \\ J_{n-1}(x) - J_{n+1}(x) &= 2J_n'(x) \end{aligned} \tag{11}$$

to prove the following formulas:

(a) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

(b) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}$

(c) $J_n x = J'_{n+1} + \frac{n+1}{x} J_{n+1}(x)$

Solution:

(a) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

$$\begin{aligned} \frac{d}{dx}[x^n J_n(x)] &= \frac{d(x^n)}{dx} J_n(x) + x^n \frac{dJ_n(x)}{dx} && (12) \\ &= nx^{n-1} J_n(x) + x^n J_n'(x) && (13) \\ &= \frac{x^n}{2}\left(\frac{2n}{x} J_n(x)\right) + \frac{x^n}{2}\left(2J_n'(x)\right) && (14) \\ &= \frac{x^n}{2}\left\{J_{n-1}(x) + J_{n+1}(x) + J_{n-1} - J_{n+1}(x)\right\} && (15) \\ &= \frac{x^n}{2} 2(J_{n-1}(x)) = x^n J_{n-1}(x) && (16) \end{aligned}$$

(b) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}$

$$\begin{aligned} \frac{d}{dx}[x^{-n} J_n(x)] &= \frac{d(x^{-1})}{dx} J_n(x) + x^{-n} \frac{dJ_n(x)}{dx} && (17) \\ &= -nx^{-n-1} J_n(x) + x^{-n} J_n'(x) && (18) \\ &= -\frac{x^{-n}}{2}\left(\frac{2n}{x} J_n(x)\right) + \frac{x^{-n}}{2} 2J_n'(x) && (19) \\ &= \frac{x^{-n}}{2}\Bigg(- J_{n-1}(x) - J_{n+1} + J_{n-1}(x) - J_{n+1}(x)(x)\Bigg) && (20) \\ &= \frac{x^{-n}}{2}(-2J_{n+1}(x)) = -x^{-n} J_{n+1}(x) && (21) \end{aligned}$$

(c) $J_n(x) = J'_{n+1}(x) + \frac{n+1}{x} J_{n+1}(x)$

$$\begin{aligned} J_{n-1}(x) + J_{n+1}(x) &= \frac{2n}{x} J_n(x) && (22) \\ J_{n-1}(x) - J_{n+1}(x) &= 2J_n'(x) && (23) \\ 2J_{n+1(x)} &= \frac{2n}{x} J_n(x) - 2J_n'(x) && (24) \\ J_{n+1(x)} &= \frac{n}{x} J_n(x) - J_n'(x) && (25) \end{aligned}$$

`14.1.5`

Derive the Jacobi-Anger expansion

$$e^{i\rho\cos\varphi} = \sum_{n=-\infty}^{\infty} i^m J_m(\rho) e^{im\varphi} \tag{26}$$

This is an expansion of a plane wave in series of cylindrical waves.

Solution:

$$e^{i\rho\cos\varphi} = \sum_{m=-\infty}^{\infty} i^m J_m(\rho) e^{im\varphi} \tag{27}$$

$$= \sum_{m=-\infty}^{\infty} J_m(\rho)(ie^{i\varphi})^m \tag{28}$$

Compare this to the generating function of Bessel equation,

$$g(x,t) = e^{(x/2)(t-1/t)} = \sum_{m=-\infty}^{\infty} J_n(x) t^n \tag{29}$$

$$\Rightarrow t = (ie^{i\varphi}); x = \rho \tag{30}$$

$$\sum_{m=-\infty}^{\infty} J_n(\rho)(ie^{i\varphi})^n = e^{(\rho/2)(ie^{i\varphi}-1/ie^{i\varphi})} \tag{31}$$

$$= e^{(\rho/2)(ie^{i\varphi}+ie^{-i\varphi})} \tag{32}$$

$$= e^{(\rho/2)2i\cos\varphi} \tag{33}$$

$$\sum_{m=-\infty}^{\infty} i^m J_m(\rho) e^{im\varphi} = e^{i\rho\cos\varphi} \tag{34}$$

`14.1.6`

Show that

(a) $\cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$

(b) $\sin x = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$

Solution:

$$\begin{aligned}
e^{i\rho\cos\varphi} &= \sum_{n=-\infty}^{\infty} i^m J_m(\rho) e^{im\varphi} && (35)\\
\Rightarrow \rho = x; \varphi = 0 & && (36)\\
e^{ix} &= \sum_{m=-\infty}^{\infty} i^m J_m(x) && (37)\\
&= J_0(x) + \sum_{m=-\infty}^{-1} (i^m) J_m(x) + \sum_{m=1}^{\infty} (i^m) J_m(x) && (38)\\
&= J_0(x) + \sum_{m=1}^{\infty} (i^{-m}) J_{-m}(x) + \sum_{m=1}^{\infty} (i^m) J_m(x) && (39)\\
&= J_0(x) + \sum_{m=1}^{\infty} (i^m) \left[(-1)^m J_{-m}(x) + J_m(x) \right] && (40)\\
&= J_0(x) + \sum_{m=1}^{\infty} (i^m) \left[(-1)^{2m} J_m(x) + J_m(x) \right] && (41)\\
&= J_0(x) + \sum_{m=1}^{\infty} 2(i^m) J_m(x) && (42)\\
&= \underbrace{\left\{ J_0(x) + 2\sum_{m=1}^{\infty} (-1)^m J_{2m}(x) \right\}}_{\cos x} + i \underbrace{\left\{ 2\sum_{m=0}^{\infty} (-1)^m J_{2m+1}(x) \right\}}_{\sin x} && (43)
\end{aligned}$$