Chapter 13 - Gamma Function

Definitions, Properties

`13.1.1`

Derive the recurrence relations

$$\Gamma(z+1) = z\Gamma(z) \tag{1}$$

from the Euler integral, Eq. (13.5),

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{2}$$

Solution: One can use integration by parts for this integral,

Sign	Derivative	Integrate
+	t^{z-1}	e^{-t}
-	$(z-1)t^{z-2}$	$-e^{-t}$

Using our integration technique, we have,

$$\int_0^\infty e^{-t} t^{z-1} dt = -t^{z-1} e^{-t}\Big|_0^\infty + (z-1)\int_0^\infty t^{z-2} e^{-t} dt \tag{3}$$

$$= (z-1)\int_0^\infty t^{z-2} e^{-t} dt \tag{4}$$

$$\Gamma(z) = (z-1)\Gamma(z-1) \tag{5}$$

we can shift this zeta value by using $z' + 1 = z$, and so we can write the equation above as,

$$\Gamma(z'+1) = z'\Gamma(z') \tag{6}$$

Since z' is just a dummy variable, we can write this as

$$\Gamma(z+1) = z\Gamma(z) \tag{7}$$

`13.1.2` unsolved

In a power-series solution for the **Legendre functions** of the second kind we encounter the expression

$$\frac{(n+1)(n+2)(n+3)\cdots(n+2s-1)(n+2s)}{2\cdot 4\cdot 6\cdot 8\cdots(2s-2)(2s)(2n+3)(2n+5)(2n+7)\cdots(2n+2s+1)} \tag{8}$$

in which s is a positive integer.

a) Rewrite this expression in terms of factorials

b) Rewrite this expression using Pochhammer symbols

`13.1.3` Note: partially solved

Show that $\Gamma(z)$ may be written

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1} dt,\ \mathcal{R}e(z) > 0 \tag{9}$$

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt,\ \mathcal{R}e(z) > 0$$

Solution: Let's start with the definition,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{10}$$

We can set, $t' = t^2$ and so we have $dt' = 2t\ dt$, then substituting this to our gamma function definition we have,

$$\Gamma(z) = \int_0^\infty e^{-t'} t'^{z-1} dt' \tag{11}$$

$$= \int_0^\infty e^{-t^2} (t^2)^{z-1} 2t dt \tag{12}$$

$$= 2\int_0^\infty e^{-t^2} t^{2z-2} t dt \tag{13}$$

$$\therefore \Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1} dt \tag{14}$$

I still don't know how to derive the second one,

`13.1.4`

In a Maxwellian distribution the fraction of particles of mass m with speed between v and $v + dv$ is

$$\frac{dN}{N} = 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right)v^2\,dv \tag{15}$$

where N is the total number of particles, k is a Boltzmann's constant, and T is the absolute temperature. The average or expectation value of v^n is defined as $< v^n >= N^{-1}\int v^n\,dN$. Show that

$$< v^n >= \left(\frac{2kT}{m}\right)^{n/2}\frac{\Gamma(\frac{n+3}{2})}{\Gamma(\frac{3}{2})} \tag{16}$$

Solution

$$\begin{aligned}
< v^n > &= N^{-1}\int v^n\,dN && (17)\\
&= \int v^n\,\frac{dN}{N} && (18)\\
&= \int 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right)v^2\,v^n\,dv && (19)\\
&= \int 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right)v^{2+n}\,dv && (20)
\end{aligned}$$

We can simplify the following equation as,

$$\begin{aligned}
< v^n > &= \int \underbrace{4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}}_{A} \exp\left(-\underbrace{\frac{m}{2kT}}_{B}v^2\right)v^{2+n}\,dv && (21)\\
&= \int A\exp(-Bv^2)v^{2+n}\ dv && (22)\\
&= A\int e^{-Bv^2}\,v^{2+n}\,dv && (23)
\end{aligned}$$

We can write $t = Bv^2, (t/B)^{1/2} = v, dt = 2Bvdv$

$$\begin{aligned}
< v^n > &= A\int e^{-Bv^2}\,v^{2+n}\,dv && (24)\\
&= A\int e^{-Bv^2}\,v\,v^n\,(vdv) && (25)\\
&= A\int e^{-t}\left(\frac{t}{B}\right)^{1/2}\left(\frac{t}{B}\right)^{n/2}\frac{dt}{2B} && (26)\\
&= \frac{A}{2B^{(\frac{n+3}{2})}}\int e^{-t}t^{\frac{n+1}{2}}\,dt && (27)
\end{aligned}$$

From the previous equation we can set,

$$z-1=\frac{n+1}{2} \tag{28}$$
$$z=\frac{n+3}{2}$$

Thus we can write the expectation value as,

$$< v^n > = \frac{A}{2B^{(\frac{n+3}{2})}}\int e^{-t}t^{\frac{n+1}{2}}\,dt \tag{29}$$
$$= \frac{A}{2B^{(\frac{n+3}{2})}}\int e^{-t}t^{z-1}dt,\ \text{for } z=\frac{n+3}{2} \tag{30}$$
$$= 2\frac{1}{\sqrt{\pi}}\left(\frac{m}{2\pi kT}\right)^{3/2}\left(\frac{2kT}{m}\right)^{\frac{n+3}{2}}\Gamma\left(\frac{n+3}{2}\right) \tag{31}$$
$$= 2\frac{1}{\sqrt{\pi}}\left(\frac{2kT}{m}\right)^{n/2}\Gamma\left(\frac{n+3}{2}\right) \tag{32}$$
$$= \frac{1}{\Gamma(3/2)}\left(\frac{2kT}{m}\right)^{n/2}\Gamma\left(\frac{n+3}{2}\right) \tag{33}$$
$$\therefore < v^n > = \left(\frac{2kT}{m}\right)^{n/2}\frac{\Gamma(\frac{n+3}{2})}{\Gamma(\frac{3}{2})} \tag{34}$$

`13.1.5`

By transforming the integral into a gamma function, show that

$$-\int_0^1 x^k \ln x\, dx = \frac{1}{(k+1)^2},\ k>-1 \tag{35}$$

Solution:

$$-\int_0^1 x^k \ln x dx = -\int_0^1 x^k \ln x dx \tag{36}$$
$$= -\int_0^1 \ln x\ x^{k+1}\frac{dx}{x} \tag{37}$$
$$\text{Let: } u=e^{(k+1)\ln x}=x^{k+1}, \ln u=(k+1)\ln x, du=(k+1)e^{(k+1)\ln u}\frac{dx}{x} \tag{38}$$
$$= -\int_0^1 \frac{\ln u}{(k+1)}\frac{du}{(k+1)} \tag{39}$$
$$= \frac{1}{(k+1)^2}\underbrace{\int_0^1 \ln u^{-1}\ du}_{\Gamma(2)},\ \text{where: } z-1=1, z=2 \tag{40}$$
$$-\int_0^1 x^k \ln x dx = \frac{1}{(k+1)^2} \tag{41}$$

`13.1.6`

Show that

$$\int_0^\infty e^{-x^4}dx = \Gamma\left(\frac{5}{4}\right) \tag{42}$$

`13.1.7`

Show that

$$\lim_{x\to 0}\frac{\Gamma(ax)}{\Gamma(x)}=\frac{1}{a} \tag{43}$$

Solution:

`13.1.8` requires knowledge in poles and residues

Locate the poles of $\Gamma(z)$. Show that they are simple poles and determines the residues.

Solution:

`13.1.9`

Show that the equation $\Gamma(x)=k, k\neq 0$, has an infinite number of real roots.

`13.1.10`

Show that, for integer s,

a)

$$\int_0^\infty x^{2s+1}\exp(-ax^2)dx=\frac{s!}{2a^{s+1}} \tag{44}$$

Solution:

$$\begin{aligned}
\int_0^\infty x^{2s+1}\exp(-ax^2)dx &= \int_0^\infty e^{-ax^2}x^{2s}xdx && (45)\\
\text{Let: } t=ax^2, dt=2axdx, \sqrt{\frac{t}{a}} &= x && (46)\\
&= \int_0^\infty e^{-t}\left(\frac{t}{a}\right)^s\frac{dt}{2a} && (47)\\
&= \frac{1}{2a^{s+1}}\int_0^\infty e^{-t}t^s dt,\ \text{we have:} z-1=s, z=s+1 && (48)\\
&= \frac{1}{2a^{s+1}}\Gamma(s+1) && (49)\\
&= \frac{s!}{2a^{s+1}} && (50)
\end{aligned}$$

b)

$$\int_0^\infty x^{2s}\exp(-ax^2)dx=\frac{\Gamma(s+\frac{1}{2})}{2a^{s+\frac{1}{2}}}=\frac{(2s-1)!!}{2^{s+1}a^s}\sqrt{\frac{\pi}{a}} \tag{51}$$

Solution:

$$\int_0^\infty x^{2s}\exp(-ax^2)dx = \int_0^\infty e^{-ax^2}x^{2s-1}xdx \tag{52}$$

$$\text{Let: } t = ax^2, dt = 2axdx, \sqrt{\frac{t}{a}} = x \tag{53}$$

$$= \int_0^\infty e^{-t}\left(\frac{t}{a}\right)^{\frac{1}{2}(2s-1)}\frac{dt}{2a} \tag{54}$$

$$= \frac{1}{2a^{s+\frac{1}{2}}}\int_0^\infty e^{-t}t^{(s-\frac{1}{2})}dt, \text{ we have:} z-1 = s-\frac{1}{2}, z = s+\frac{1}{2} \tag{55}$$

$$= \frac{1}{2a^{s+\frac{1}{2}}}\Gamma(s+\frac{1}{2}) \tag{56}$$

$$\int_0^\infty x^{2s}\exp(-ax^2)dx = \frac{\Gamma(s+\frac{1}{2})}{2a^{s+\frac{1}{2}}} \tag{57}$$

These Gaussian integrals are of major importance in statistical mechanics.

`13.1.11`

Express the coefficients of the nth term of the expansion of $(1+x)^{1/2}$ in powers of x

a) in terms of factorial of integers

b) in terms of the double factorial (!!) functions

`13.1.12`

Express the coefficients of the nth term of the expansion of $(1+x)^{-1/2}$ in powers of x

a) in terms of factorial of integers

b) in terms of the double factorial (!!) functions

`13.1.14`

a) Show that

Digamma and Polygamma Functions