Statistical Field Theories: Exactly Solved Models

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1 Exactly Solved Models

1.1 Luttinger 1-D model of interacting systems

Luttinger model is a model of interacting fermions, assumed to be massless, in one-dimension. It is unrealistic, but it has realistic features as well such as the true pair interaction between the particles.

1.2 Tomonaga-Luttinger liquids

1.3 Ising spin lattice model

It was invented by Wilhelm Lenz and was given to his student, Ernst Ising. 1D problem was solved by Ising himself in his 1924 thesis. No phase transition exist for 1D but for 2D square lattice, Ising model is much harder. Twenty years later, Lars Onsager solved it using transfer-matrix method. Other approaches of solving 2D Ising model are related to quantum field theory. The 3D Ising model is best described by a conformal field theory. In dimensions greater than 4, the phase transition of the Ising model is described by mean field theory.

1.3.1 The Problem

Consider a lattice sites L, each with a set of adjacent sites forming a d-dimensional lattice. For each lattice site $k \in L$ their is a discrete variable σ_k such that $\sigma_k \in \{+1, -1\}$. A spin configuration, $\sigma = (\sigma_k)_{k \in L}$, is an assignment of spin value to each lattice site.

Two adjacent sites, $i, j \in L$ there is an interaction J_{ij}. A site $j \in L$ has an external magnetic field h_j interacting with it. The energy of a configuration σ is given by the Hamiltonian function,

$$H(\sigma) = -\sum_{<ij>} J_{ij}\sigma_i\sigma_j - \mu \sum_j h_j \sigma_j \tag{1}$$

Ising model can be classified according to the sign of the interaction:

1. $J_{ij} > 0$, the interaction is ferromagnetic, spins desire to be aligned

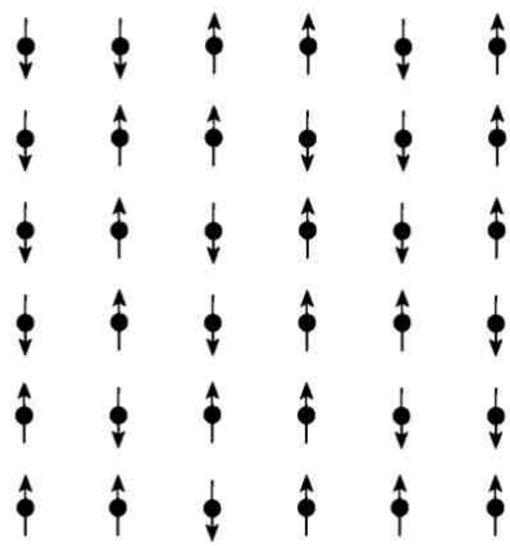

Figure 1: A lattice site with its spin configuration.

2. $J_{ij} < 0$, the interaction is antiferromagnetic, adjacent spins tend to have opposite signs

3. $J_{ij} = 0$, the spins are noninteracting

How the spins interact with the external field, is dictated by the sign of the h_j

1. $h_j > 0$, the spin site j desires to line up in the positive direction

2. $h_j < 0$, the spin site j desires to line up in the negative direction

3. $h_j = 0$, there are no external influence on the spin site

Special Case: Ising models are often examined without an internal field, that is, $h_j = 0\ \forall j \in L$, and assume that all nearest neighbors $< ij >$ have the same interaction strength $J = J_{ij}\ \forall i, j \in L$. From this simplification, the Hamiltonian is reduced to,

$$H(\sigma) = -J \sum_{<ij>} \sigma_i \sigma_j \tag{2}$$

1.3.2 Square Lattice

The model itself is a square lattice, each cell containing a value of either "1" or "-1", representing an electron's spin. Each spin interacts only with its nearest neighbours.

1.3.3 2D-Ising model in python:

The main code creates a lattice of randomly arranged spins at a given temperature. The spins then either flip or not by calculating the energy difference ΔU between the considered spin and its 4 nearest neighbors using the formula:

$$\Delta U = 2J * spin[i][j](spin_{left} + spin_{right} + spin_{top} + spin_{bottom}) \tag{3}$$

We perform l iteration until it reaches thermal equilibrium. Then another $l/2$ iterations to measure the necessary lattice properties.

From the model, we will examine the following lattice characteristics:

1. Energy per site:

$$E_{spin} = -J \cdot spin[i][j](spin_{left} + spin_{right} + spin_{top} + spin_{bottom}) \tag{4}$$

2. Magnetization per site

$$spin = \frac{\Delta M}{size \cdot size} \tag{5}$$

3. Magnetic susceptibility:

$$\chi = \frac{1}{T}[<M^2> - <M>^2] \tag{6}$$

4. Specific Heat

$$C_V = \frac{1}{T^2}[<E^2> - <E>^2] \tag{7}$$

5. Correlation Function

$$G_{spin}(r) =< spin(r)spin(0) > - < spin >^2 \tag{8}$$

6. Correlation length

1.3.4 Solution to the Ising chain in terms of Majorana Fermions

1.3.5 Solution to the 2D Ising model via Kramers-Wannier Duality

1.4 Transverse Ising model

1.5 Kitaev Honeycomb Lattice Model

The model consists of a honeycomb lattice with a spin sitting on each of its vertices. The spins interact with their nearest neighbors via three different types of links. The x-, y- and z-interaction. The interaction of the spins can be described using the pauli spin operators σ_i^α where $\alpha = x, y, z$ and i indexing the site.