Mathematical Methods for Physicists

Chapter 7: ODE

Many equations in theoretical physics are originally formulated in terms of differential equations in the the 3D physical space (and sometimes also time).

These variables, such as the x, y, z and t are referred to as **independent variables**, while the functions being differentiated are referred to as **dependent variables.**

Differential equation involving more than one independent variable is called a **partial differential equation (PDE)**. The first section will deal with an equation in a single independent variable, they're known as an **ordinary differential equation (ODE)**.

As we shall see later, some of the most frequently used methods for solving PDEs involve expressions in terms of the solutions to ODEs, so it is appropriate to master ODE first before one proceeds to PDEs.

Introduction

Linearity of the Operator $\mathcal{L} = \frac{d}{dx}$ **and Higher Derivatives** Derivative is a linear operation, by that we mean that

$$\frac{d}{dx}\big(a\varphi(x) + b\psi(x)\big) = a\frac{d\varphi}{dx} + b\frac{d\psi}{dx}$$

and the derivative operation can be viewed as a linear operator: $\mathcal{L} = \frac{d}{dx}$ Higher derivatives are linear operators as well,

$$\frac{d^2}{dx^2}\big(a\varphi(x) + b\psi(x)\big) = a\frac{d^2\varphi}{dx^2} + b\frac{d^2\psi}{dx^2}$$

The **general form of an operator** is given by,

$$\mathcal{L} \equiv \sum_{\nu=0}^{n} p_v(x)\left(\frac{d^\nu}{dx^\nu}\right)$$

where the function $p_\nu(x)$ are arbitrary.

For example, if the operator is defined as,

$$\mathcal{L} = p(x)\frac{d}{dx} + q(x)$$

Then its linearity can be shown as follows,

$$\begin{aligned}\mathcal{L}(a\varphi(x) + b\psi(x)) &= a\left(p(x)\frac{d}{dx} + q(x)\right) + b\left(p(x)\frac{d}{dx} + q(x)\right)\\ &= a\mathcal{L}\varphi + b\mathcal{L}\psi\end{aligned}$$

Homogeneity and Linearity The ODE is called **homogenous** if the dependent variable occurs to the same power in all its terms, otherwise it is called **inhomogeneous**. It is linear if it can be written in the form,

$$\mathcal{L}\varphi(x) = F(x)$$

Where $\mathcal{L}$ is some operator and $F(x)$ is an algebraic function of x *(not a differential operator)*. An important class of ODEs are those that are both linear and homogeneous, and thereby of the form $\mathcal{L}\varphi(x) = 0$

Linear Dependence ODEs solutions are in general not unique. There are instances that multiple solutions exist, and for this kind of cases, it is useful to identify those that are linearly independent.

Homogeneous linear ODEs have the general property that any multiple of the solutions is also a solution. If there are multiple linearly independent solutions, any linear combination of those solutions will also solve the ODE.

Consider the following solutions phi and psi:

$$\mathcal{L}\varphi = 0 \text{ and } \mathcal{L}\psi = 0 \rightarrow \mathcal{L}(a\varphi + b\psi) = 0$$

Examples of ODE

Schrodinger equation of Quantum Mechanics S.E. is a homogenous linear ODE and the property that any linear combination of its solutions is also a solution is the basis of the well-known superposition principle in electrodynamics, wave optics, and quantum theory.

Some physically important problems give rise to nonlinear differential equations

1. Bernoulli equation
   . ?
   . ?

Other terms used to classify an ODE include:

- **Order** – higher derivative appearing
- **Degree** – power to which the highest derivative appears after the ODE is rationalized if that is necessary

First-Order Equations

Many physical systems can be studied using *first-order differential equations.* We consider the general form:

$$\frac{dy}{dx} = f(x, y) = -\frac{P(x, y)}{Q(x, y)}$$

There is not systematic way to solve first-order equations, however, there are techniques that are often useful. Here is the list of the techniques:

- Separable equations
- Exact differentials
- Equations homogenous in x and y
- Isobaric equations

After that, we will have a detailed treatment of linear first-order ODEs, for which systematic procedures are available.

Techniques

1. Separable Equations

It is given in the form:

$$\frac{dy}{dx} = -\frac{P(x)}{Q(y)}$$

which can be rewritten as

$$P(x)dx + Q(y)dy = 0$$

Integrating from given points (x_0, y_0) to (x, y) yields

$$\int_{x_0}^{x} P(x)dx + \int_{y_0}^{y} Q(y)dy = 0$$

2. Exact Differentials

Now, consider the function P and Q as a function of x and y.

$$P(x,y)dx + Q(x,y)dy = 0$$

This equation is said to be **exact** if we can match the left-hand side of it to a differential, let say $d\varphi$, and thereby reach

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = 0$$

One can view it this way by comparing the following equations:

$$\begin{aligned} P(x,y)dx + Q(x,y)dy = 0 \\ \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = 0 \end{aligned}$$

Exactness implies that there exists a function $\varphi(x,y)$ such that,

$$\frac{\partial \varphi}{\partial x} = P(x,y) \text{ and } \frac{\partial \varphi}{\partial y} = Q(x,y)$$

A simple condition to determine if the differential equation is exact is the following equality:

$$\frac{\partial P(x,y)}{\partial x} = \frac{\partial Q(x,y)}{\partial y}$$

If this is the case, then our solutions is the integration of the differential of psi which is $\varphi(x,y) = constant$.

What if the condition is not satisfied?

We need to find what is called the integrating factors $\alpha(x,y)$ such that

$$\alpha(x,y)P(x,y)dx + \alpha(x,y)Q(x,y)dy = 0$$

Unfortunately, an integrating factor is not always easy to find. We'll encounter a systematic way to develop an integrating factor when the first-order ODE is linear. Note also that an integrating factor always exists.

3. Equations Homogenous in x and y

If the ODE is written as this,

$$P(x,y)dx + Q(x,y)dy = 0$$

It is said to be *homogenous (of order n)* in x and y if the combined powers of x and y *adds up* to **n** in all terms of $P(x,y)$ and Q(x,y).

For this type of differential equations, first-order ODE can be made by the substitution $y = xv$ with $dy = xdv + vdx$.

This substitution causes the x dependence of all the terms of the equation containing dv to be x^{n+1}, with all the terms containing dx having x-dependence x^n. The variables x and dv can then be separated.

4. Isobaric Equations

From previous cases, we made an assumption that $y = xv$, now we can introduce a generalization by modifying different weights to x and y: $y = x^m v$. This will make our equations separable.

Linear First-order ODEs

While nonlinear first-order ODEs can often be solved using the strategies already presented (1-4), the situation for *linear first-order ODE* is different. A specific procedure exists for solving the general equation of this type, which is in the form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

If it is exact, its solution is straightforward. If not exact, we introduce the integrating factors $\alpha(x)$, so that our ODE now is,

$$\alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y = \alpha(x)q(x)$$

We want actually the left-hand side to be a perfect differential:

$$\begin{aligned} \frac{d}{dx}[\alpha(x)y] &= \alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y \\ \alpha(x)\frac{dy}{dx} + \frac{d\alpha(x)}{dx}y &= \alpha(x)\frac{dy}{dx} + \alpha(x)p(x)y \end{aligned}$$

so α must satisfy

$$\frac{d\alpha}{dx} = \alpha(x)p(x)$$

This is a separable equation so a solution exists, in which separating the variables and integrating, we obtain:

$$\int^{\alpha} \frac{d\alpha}{\alpha} = \int^{x} p(x)dx$$

Upon completing the evaluation we have,

$$\alpha(x) = exp\left[\int^{x} p(x)dx\right]$$

We now know , so we proceed to integrate the ODE which is now simplified to this equation:

$$\frac{d}{dx}\left[\alpha(x)y(x)\right] = \alpha(x)q(x)$$

which yields the solution,

$$y(x) = \int^x \alpha(x) q(x) dx + C \equiv y_2(x) + y_1(x)$$

Let's examine then the solution of a linear first-order ODE.

1. If $q(x)$ is zero the general solution of this homogenous equation is in terms of $y_1 = C/\alpha(x)$
   . The other term, $y_2 = \frac{1}{\alpha(x)} \int^x \alpha(x) q(x) dx$, corresponding to the **source term** $q(x)$ and is a solution of the original inhomogeneous equation.

We thus have a general solution to an inhomogeneous equation presented as a **particular solution** plus the general solution to the corresponding homogenous equation.

We now present important theorems about our results:

> The solution of an inhomogeneous first-order linear ODE is unique except for an arbitrary multiple of the solution of the corresponding homogeneous ODE.

We also have the theorem:

> A first-order linear homogenous ODE has only one linearly independent solution

ODEs with Constant Coefficients

There is a class of ODE that is frequently occurs that are not constrained of specific order, namely those that are linear and whose homogeneous terms have constant coefficients.

Generic equation is of type:

$$\frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = F(x)$$

This generic equation has solutions of the form $y = e^{mx}$, where m is a solution of the algebraic equation

$$m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

If the m equation has a multiple root, the above prescription will not yield the full set of n linearly independent solutions for the original nth order ODE.

Note, if e^{mx} is a solution, then so is $\frac{de^{mx}}{dm} = xe^{mx}$. The triple root would have solutions $e^{mx}, xe^{mx}, x^2 e^{mx}$.

Second-Order Linear ODEs

The main topic of this chapter is the Second-order linear ODEs. These are of particular importance because they arise in the most frequently used methods for solving PDEs in quantum mechanics, electromagnetic theory, and other areas in physics.

Unlike first-order linear ODEs there is no applicable closed-form solution, and in general it is found advisable to use methods that produce solutions in the form of power series.

As precursor to the discussion of series-solution methods, we begin by examining the notion of singularity as applied to ODEs.

Singular Points

Singularity concept in ODE is important for two reasons:

1. it is useful for classifying ODEs and identifying those that can be transformed into common forms
2. it bears on the feasibility of finding series solutions to the ODE (the topic of Fuchs' theorem)

The **linear homogeneous second-order ODE** is written in the form:

$$y'' + P(x)y' + Q(x)y = 0$$

Analyzing at $x_0 \to 0$

A point x_0 for which $P(x)$ and $Q(x)$ are finite are termed **ordinary points** of the ODE.

However, if either $P(x)$ or $Q(x)$ diverge as $x \to x_0$, the point x_0 is called a **singular point**.

Classification of Singular Points

1. **Regular points**: if either $P(x)$ or $Q(x)$ diverges there, but $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ remain finite.
2. **Irregular points**: if $P(x)$ diverges faster than $1/(x - x_0)$ so that $(x - x_0)P(x)$ goes to infinity as $x \to x_0$, or if $Q(x)$ diverges faster than $1/(x - x_0)^2$ so that $(x - x_0)^2 Q(x)$ goes to infinity as $x \to x_0$.

Analyzing at $x_0 \to \infty$

To analyze the behavior at $x \to \infty$, we set $x = \frac{1}{z}$, substitute into the differential equation, and examine the behavior in the limit $z \to 0$.

The ODE, originally in the dependent variable $y(x)$ will now be written in terms of $w(z)$, defined as $w(z) = y(z^{-1})$.

Converting the derivatives,

$$y' = \frac{dy(x)}{dx} = \frac{dy(z^{-1})}{dz}\frac{dz}{dx} = \frac{dw(z)}{dz}\left(\frac{-1}{x^2}\right) = -z^2 w'$$

Similarly,

$$\begin{aligned} y'' &= \frac{dy'}{dz}\frac{dz}{dx} \\ &= \frac{d}{dz}(-z^2 w')\frac{dz}{dx} \\ &= (-2zw')\frac{dz}{dx} - z^2 w''\frac{dz}{dx} \\ &= (-2zw')(-z^2) - z^2 w''(-z^2) \\ &= 2z^3 w' + z^4 w'' \end{aligned}$$

We can then transform our linear homogeneous second-order ODE into:

$$
\begin{aligned}
0 &= y'' + P(x)y' + Q(x)y \\
0 &= z^4 w'' + 2z^3 w' + P(z^{-1})(-z^2 w') + Q(z^{-1})w \\
0 &= z^4 w'' + [2z^3 - P(z^{-1})z^2]w' + Q(z^{-1})w \\
0 &= w'' + \frac{2z^3 - P(z^{-1})z^2}{z^4} w' + \frac{Q(z^{-1})}{z^4} w \\
0 &= w'' + \frac{2z - P(z^{-1})}{z^2} w' + \frac{Q(z^{-1})}{z^4} w
\end{aligned}
$$

Thus, we can see that the possibility of a singularity at $z = 0$ depends on the behavior of

$$\frac{2z - P(z^{-1})}{z^2} \text{ and } \frac{Q(z^{-1})}{z^4}$$

Important case:

- If these two expressions remain finite at $z = 0$, the point $x = \infty$ is an ordinary point.
- If they diverge no more rapidly than $\frac{1}{z}$ and $\frac{1}{z^2}$, respectively, $x = \infty$ is a regular singular point
- otherwise it is an irregular singular point (an essential singularity).

Consider the following example:

Bessel's Equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

We compare this to linear homogenous second-order equation:

$$
\begin{aligned}
x^2 y'' + xy' + (x^2 - n^2)y &= 0 \\
y'' + \frac{1}{x} y' + \frac{(x^2 - n^2)}{x^2} y &= 0 \\
y'' + P(x)y' + Q(x)y &= 0
\end{aligned}
$$

we have,

$$P(x) = \frac{1}{x}, Q(x) = 1 - \frac{n^2}{x^2}$$

which shows that at $x = 0$ is a regular singularity.

How is this that case?

We have to recall the previous definition of a singular point. We say that if either $P(x)$ or $Q(x)$ diverges as $x \to x_0$ then x_0 is said to be a singular point. Note that in Bessel's equation our singular point is at $x_0 = 0$, since $P(x) \to \infty$ at $x \to 0$. We just need one coefficient to diverge.

To categorize our singular point $x_0 = 0$ to be a **regular or irregular point** we have to refer to our definition. We know now that either $P(x)$ or $Q(x)$ diverges, now we need know if $(x - x_0)P(x)$ or $(x - x_0)^2 Q(x)$ remains finite.

Now, lets look at the scaled coefficients:

$$
\begin{aligned}
(x-x_0)P(x) &= (x-0)P(x) \\
&= xP(x) \\
&= x(\frac{1}{x}) \\
&= 1
\end{aligned}
\qquad
\begin{aligned}
(x-x_0)^2Q(x) &= (x-0)^2Q(x) \\
&= x^2Q(x) \\
&= x^2\left(1-\frac{n^2}{x^2}\right) \\
&= x^2-n^2 = -n^2 \qquad (x\to 0)
\end{aligned}
$$

It is evident that both remains finite even if x_0 is a singular point. Thus we can say that at $x=0$ it is a regular singularity. There are no other singularities in the finite range.

Now, let's consider the $x\to\infty(z\to 0)$. Using the expression of $P(x)$ and $Q(x)$, and the coefficients when $x\to\infty$, we have

$$
\begin{aligned}
&= \frac{2z-P(z^{-1})}{z^2} \\
&= \frac{2z-z}{z^2} \\
&= \frac{1}{z}
\end{aligned}
\qquad
\begin{aligned}
&= \frac{Q(z^{-1})}{z^4} \\
&= \frac{1-n^2z^2}{z^4} \\
&= \frac{1}{z^4}-\frac{n^2}{z^2}
\end{aligned}
$$

It is apparent from the coefficients that the expressions diverges as $z\to 0$, in which no more rapidly than $\frac{1}{z}$ and $\frac{1}{z^4}$. Hence, the point $x=\infty$ is an irregular, or essential singularity.

Let's consider another example.

Linear Harmonic oscillator

$$
y''+\omega^2y=0
$$

From this equation it is easy to see that $P(x)=\omega^2$ and $Q(x)=0$. Thus the point $x=0$ is not a singular point. However at $x\to\infty$, we have the following coefficients,

$$
\begin{aligned}
&= \frac{2z-P(z^{-1})}{z^2} \\
&= \frac{2z-\omega^2}{z^2} \\
&= \frac{2}{z}-\frac{\omega^2}{z^2}
\end{aligned}
\qquad
\begin{aligned}
&= \frac{Q(z^{-1})}{z^4} \\
&= 0 \\
&= 0
\end{aligned}
$$

Now the coefficients is obviously diverging no more rapidly than $\frac{1}{z^2}$ for the $P(x)$. Thus we can say that $x=\infty$ is an irregular singular point.

The following table lists the singular points of a number of ODEs of importance in physics.

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The first three equation has 3 regular singular points.

Let's try to answer some exercises in the book.

Exercises

`7.4.1` Show that Legendre's equation has regular singularities at $x=-1,1,$ and ∞.

Let's begin with the Legendre's equation:

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$

Our first goal is to rewrite the equation above as linear homogeneous second-order DE,

$$0 = y'' + P(x)y' + Q(x)y$$

To do this we divide the Legendre's equation with the coefficient of the y''. Thus we have,

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$
$$y'' + \underbrace{\Big\{-\frac{2x}{(1-x^2)}\Big\}}_{P(x)}y' + \underbrace{\frac{l(l+1)}{(1-x^2)}}_{Q(x)}y = 0$$

1. First step is to analyze at what point our $P(x)$ or $Q(x)$ will be singular in a finite range. Apparently the coefficient, $1-x^2 = (1-x)(1+x)$ which gives us two possible singular point for $P(x)$ and $Q(x)$ that is ± 1.
   - At $x=1$, we have $(x-1)P(x) = -\frac{2x}{(x+1)}$ which is finite as $x \to 1$. Similarly for $(x-1)^2 Q(x) = \frac{l(l+1)(1-x)}{(1+x)}$ which is also finite as $x \to 1$. We can conclude then that $x=1$ is a regular singular point
   - At $x=-1$, we have $(x+1)P(x) = -\frac{2x}{(x-1)}$ which is finite as $x \to -1$. Similarly for $(x+1)^2 Q(x) = \frac{l(l+1)(1+x)}{(1-x)}$ which is also finite as $x \to -1$. We can conclude then that $x=-1$ is a regular singular point
2. Second step is to analyze if at infinity our $P(x)$ or $Q(x)$ will diverge. We have to express our coefficients first in terms of this relation $x = \frac{1}{z}$.
   - $P(x) \to P(z^{-1})$ so we have $-\frac{2x}{(1-x^2)} \to -\frac{2(1/z)}{(1-(1/z)^2)} = \frac{2z}{1-z^2}$ similarly, we have $Q(x) \to Q(z^{-1})$ and we have $\frac{l(l+1)}{(1-x^2)} \to \frac{z^2 l(l-1)}{z^2-1}$. Clearly both diverges as $x \to \infty$.

Series Solutions – Frobenius' Method

In this section, we will develop a method of obtaining solution(s) of the linear, second-order, homogenous ODE.

But first, we will develop the mechanics of the method. We will study some basic examples and return to discuss the condition under which we can expect these series solutions to exist.

Again, consider the homogenous linear, second-order ODE of the form,

$$y'' + P(x)y' + Q(x)y = 0$$

We can develop a solution by expansions about the point $x=0$. In the next section, we develop **second, independent solution and prove that no third, independent solution exists.**

The most general solution of the equation above is in terms of two independent solutions,

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

What if the right hand side is not zero? (an inquisitive student might ask!)

$$y'' + P(x)y' + Q(x)y = F(x)$$

The function on the right, $F(x)$, typically represents a source (such as electrostatic charge or a driving force). Methods for solving this inhomogeneous ODE will be discuss in later chapters (Chapter 20) using Laplace transform techniques.

If the inhomogeneous ODE has a **particular solution** y_p, we may add to it any solution corresponding to the homogeneous equation, and write the most general solution as ,

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

The constant coefficients c_1 and c_2 will be fixed by boundary conditions.

For the meantime, we shall deal with second order ODE that is homogeneous, $F(x) = 0$. We will develop a solution by substituting into it a power series with undetermined coefficients. *Also available as a parameter is the power of the lowest nonvanishing term of the series*. (?)

`First Example - Linear Oscillator`

Consider the linear oscillator equation:

$$\frac{d^2 y}{dx^2} + \omega^2 y = 0$$

(we've solved this in previous section and showed that the solution were $y = \sin \omega x$ and $\cos \omega x$). We try the following series,

$$\begin{aligned} y(x) &= x^s (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \\ &= \sum_{j=0} a_j x^{s+j}, a_0 \neq 0 \end{aligned}$$

with the coefficients a_j and exponent s still undetermined. Note that s need not be an integer.

By differentiation we have,

$$\frac{d^2 y}{dx^2} = \sum_{j=0} a_j (s+j)(s+j-1) x^{s+j-2}$$

Now we substitute the these series equations to our ODE, we have,

$$\begin{aligned} \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} + \omega^2 \sum_{j=0}^{\infty} a_j x^{s+j} &= 0 \\ a_0 s(s-1) x^{s-2} + a_1 s(s+1) x^{s-1} + \sum_{j=2}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} + \omega^2 \sum_{j=0}^{\infty} a_j x^{s+j} &= 0 \text{ (j'=j+2)} \\ a_0 s(s-1) x^{s-2} + a_1 s(s+1) x^{s-1} + \sum_{j'+2=2}^{\infty} a_j (s+j'+2)(s+j'+1) x^{s+j'} + \omega^2 \sum_{j=0}^{\infty} a_j x^{s+j} &= 0 \\ a_0 s(s-1) x^{s-2} + a_1 s(s+1) x^{s-1} + \sum_{j=0}^{\infty} \left\{ a_{j+2}(s+j+2)(s+j+1) + \omega^2 a_j \right\} x^{s+j} &= 0 \end{aligned}$$

The lowest power of x is x^{s-2} occurring only for $j=0$ in the first summation. By definition, $a_0 \neq 0$, so for the lowest power of x, the coefficients is expressed now as,

$$\begin{aligned} a_0 s(s-1) &= 0 \qquad \text{since } a_0 \neq 0 \text{ we have} \\ s(s-1) &= 0 \end{aligned}$$

This equation, coming from the coefficients of the lowest power of x, is called the indicial equation. The indicial equation and its roots are of critical importance to our analysis. In our previous equation, it informs us that either $s=0$ or $s=1$, so that our series solution must start either at x^0 or an x^1 term.

The next lowest power of x, namely x^{s-1} also occurs uniquely for $j=1$ in the first summation. Setting the coefficient of x^{s-1} to zero, we have

$$a_1(s+1)s = 0$$

This shows that if $s=1$ we must have $a_1=0$. However, if $s=0$, this equation imposes no requirement on the coefficient set.

Looking at the final equation in the series equation we have above, the third equation we have is given as,

$$\begin{aligned} 0 &= a_{j+2}(s+j+2)(s+j+1) + \omega^2 a_j \\ a_{j+2} &= -a_j \frac{\omega^2}{(s+j+2)(s+j+1)} \end{aligned}$$

This is a two-term **recurrence relation**. Given a_j it allows us to compute for a_{j+2}, and then a_{j+4}, a_{j+6}, and so on.

If we start with a_0 we can compute for the even coefficients, but we have no information for the odd coefficients. Since a_1 is arbitrary if $s=0$ and necessarily zero if $s=1$, let us set it equal to zero, and then we have,

$$a_3 = a_5 = a_7 = \cdots = 0$$

The result is that all odd-numbered coefficients vanish.

For `s=0` , the recurrence equation becomes,

$$a_{j+2} = -a_j \frac{\omega^2}{(j+2)(j+1)}$$

This recurrence equations gives us,

$$\begin{aligned} a_2 &= -a_0 \frac{\omega^2}{2 \cdot 1} = -a_0 \frac{\omega^2}{2!} \\ a_4 &= -a_2 \frac{\omega^2}{4 \cdot 3} = a_0 \frac{\omega^4}{4!} \\ a_6 &= -a_4 \frac{\omega^2}{6 \cdot 5} = -a_0 \frac{\omega^6}{6!} \\ &\vdots \end{aligned}$$

By inspection, we can express the following coefficients as,

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{2n!} a_0$$

and our solution is,

$$\begin{aligned}
y(x) &= x^s\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) \\
&= \sum_{j=0} a_j x^{s+j},\, a_0 \neq 0 && @s = 0 \\
&= x^0 a_0 \left(1 - \frac{\omega^2}{2!}x^2 + \frac{\omega^4}{4!}x^4 - \frac{\omega^6}{6!}x^6 + \cdots\right) \\
y(x) &= a_0\left(1 - \frac{(\omega x)^2}{2!} + \frac{(\omega x)^4}{4!} - \frac{(\omega x)^6}{6!} + \cdots\right) \\
y(x) &= a_0 \cos \omega x
\end{aligned}$$

Similarly, for `s=1`, the recurrence equation becomes,

$$a_{j+2} = -a_j \frac{\omega^2}{(j+3)(j+2)}$$

This recurrence equations gives us,

$$\begin{aligned}
a_2 &= -a_0 \frac{\omega^2}{3 \cdot 2} = -a_0 \frac{\omega^2}{3!} \\
a_4 &= -a_2 \frac{\omega^2}{5 \cdot 4} = a_0 \frac{\omega^4}{5!} \\
a_6 &= -a_4 \frac{\omega^2}{7 \cdot 6} = -a_0 \frac{\omega^6}{7!} \\
&\vdots
\end{aligned}$$

By inspection, we can express the following coefficients as,

$$a_{2n} = (-1)^n \frac{\omega^{2n}}{(2n+1)!} a_0$$

For s=1, we obtain the following solution,

$$\begin{aligned}
y(x) &= x^s\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) \\
&= \sum_{j=0} a_j x^{s+j},\, a_0 \neq 0 && @s = 0 \\
&= x^1 a_0 \left(1 - \frac{\omega^2}{3!}x^2 + \frac{\omega^4}{5!}x^4 - \frac{\omega^6}{7!}x^6 + \cdots\right) \\
&= a_0 \left(1 - \frac{\omega^2}{3!}x^3 + \frac{\omega^4}{5!}x^5 - \frac{\omega^6}{7!}x^7 + \cdots\right) \\
y(x) &= \frac{a_0}{\omega}\left(1 - \frac{(\omega x)^3}{3!} + \frac{(\omega x)^5}{5!} - \frac{(\omega x)^7}{7!} + \cdots\right) \\
y(x) &= \frac{a_0}{\omega} \sin \omega x
\end{aligned}$$

We note, for future reference, that the ODE solution from the indicial equation root $s = 0$ consisted only of even powers of x, while the solution from the root $s = 1$ contained only odd powers.

To summarize this approach, we may write schematically as shown below:

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From the uniqueness of power series (Section 1.2) the total coefficient of each power of x mush vanish -all by itself.

> The indicial equation gives us the first coefficient that vanishes as in (I). The second coefficient is handled by setting $a_1 = 0$ in (II). The vanishing values of the coefficients of x^s is ensured by imposing the recurrence equation in (III) and (IV).

Frobenius' method gives us two series solutions of the linear oscillator equation. However there are two points about such series solutions that must be strongly emphasized:

1. The series solution should always be substituted back into the differential equation to see if it works. If it works, it is a solution
2. The acceptability of a series solution depends on its convergence. It is possible for a solution of Frobenius' method to satisfies the original differential equation but does not converge over the region of interest.

Expansion about x_0

Note that this equation

$$y(x) = \sum_{j=0} a_j x^{s+j}, a_0 \neq 0$$

is an expansion about the origin $x_0 = 0$. It is possible to replace with a series that is expanded about some number x_0.

$$y(x) = \sum_{j=0} a_j (x - x_0)^{s+j}, a_0 \neq 0$$

For the Legendre, Chebyshev, and hypergeometric equations, the choice $x_0 = 1$ has some advantages.

- The point x_0 **should not be chosen at an essential singularity** (irregular singular point) or Frobenius' method will probably fail.
- if x_0 is an **ordinary point or regular singular point** the resultant series will be valid where it converges.
    - you can expect a divergence of some sort when $|x - x_0| = |z_1 - x_0|$, where z_1 is the ODE's closest singularity to x_0 (in the complex plane) (??)

Symmetry of Solutions

Note that the solution of the classical oscillator problem we obtained has even symmetry, $y_1(x) = y_1(-x)$, and one odd symmetry $y_2(x) = -y_2(-x)$.

> This is not just an accident but a direct consequence of the form of the ODE.

The general homogeneous ODE is written as,

$$\mathcal{L}(x)y(x) = 0$$

in which $\mathcal{L}$ is the differential operator, we see that for the linear oscillator equation, $\mathcal{L}(x)$ is even under parity; that is,

$$\mathcal{L}(x) = \mathcal{L}(-x)$$

- whenever the operator has parity or symmetry (even or odd), we may interchange (+x) and (-x), and our general homogeneous ODE becomes

$$\begin{aligned} \mathcal{L}(x)y(x) &= 0 & x &\to -x \\ \mathcal{L}(-x)y(-x) &= 0 & \text{for the parity part: } \mathcal{L}(x) &= \pm\mathcal{L}(-x) \\ \pm\mathcal{L}(x)y(-x) &= 0 \end{aligned}$$

- If $y(x)$ is a solution of the differential equation, y(-x) is also a solution. Then either $y(x)$ and $y(-x)$:
    - they are linearly dependent, i.e. proportional meaning that y is either even or odd,

    $$y = c_1 y_{even}(x) + c_2 y_{odd}(x)$$

    - they are linearly independent solutions that can be combined into a pair of solutions.

    $$y_{even} = y(x) + y(-x) \qquad y_{odd} = y(x) - y(-x)$$

Our method for finding the solution for the classical oscillator leads us to the symmetry of the solution of the ODE.

Note that **Legendre, Chebyshev, Bessel, simple harmonic oscillator and Hermite equations** are based on differential equation with even parity.

For example Bessel's equation is given with the following operator:

$$\begin{aligned} 0 &= x^2 \frac{d^2}{dx^2} y + x \frac{d}{dx} y + (x^2 - n^2) y \\ 0 &= \left\{ x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + (x^2 - n^2) \right\} y \end{aligned}$$

We can show the parity of its operator as follows,

$$\mathcal{L}(x) = x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + (x^2 - n^2) \qquad \begin{aligned} \mathcal{L}(-x) &= (-x)^2 \frac{d^2}{d(-x)^2} + (-x)\{\frac{d}{-dx}\} + ((-x)^2 - n^2) \\ \mathcal{L}(-x) &= (x)^2 \frac{d^2}{d(x)^2} + x\{\frac{d}{dx}\} + (x^2 - n^2) \end{aligned}$$

$$\mathcal{L}(x) = \mathcal{L}(-x)$$

There is another way to show even parity, that is, their $P(x)$ is odd and $Q(x)$ is even. *(This comes from the fact that y' is odd so P(x) has to be odd, on the other hand Q(x) is not operated at all, hence it is left to be even.)*

Solutions of all of them may be represented as series of even powers of x or separate series of odd powers of x..

Laguerre differential operator has neither even nor odd symmetry; hence its solutions cannot be expected to exhibit even or odd parity.

> Parity is emphasize here because it is important in quantum mechanics. We find that in may problems wave functions are either even or odd, meaning that they have a definite parity. Most interactions are also even or odd, and the result is that parity is conserved.

`Second Example - Bessel's Equation`

In our first example, we obtained two solutions with no trouble ta all.

To get some idea of other things that can happen, we try to solve Bessel's equation,

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

Again, we assume that solution of the form,

$$y(x) = \sum_{j=0}^{\infty} a_j x^{s+j}$$

We then take its first and second derivative,

$$\begin{aligned} y(x) &= \sum_{j=0}^{\infty} a_j x^{s+j} \\ y'(x) &= \sum_{j=0}^{\infty} a_j (s+j) x^{s+j-1} \\ y''(x) &= \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} \end{aligned}$$

Substitute this into our Bessel's equation, the result is,

$$\begin{aligned} x^2\left(\sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2}\right) + x\left(\sum_{j=0}^{\infty} a_j (s+j) x^{s+j-1}\right) \\ +(x^2 - n^2)\sum_{j=0}^{\infty} a_j x^{s+j} = 0 \end{aligned}$$

Simplifying the equation above, we have:

$$\begin{aligned} \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j} + \sum_{j=0}^{\infty} a_j (s+j) x^{s+j} \\ + \sum_{j=0}^{\infty} a_j x^{s+j+2} - \sum_{j=0}^{\infty} a_j n^2 x^{s+j} = 0 \end{aligned}$$

For the third term of expression above, we can set a dummy variable $j = j' - 2$.

$$\cdots + \sum_{j'=2}^{\infty} a_{j'-2} x^{s+j'} +$$

We then need to shift the three other terms to start at $j = 2$.

$$a_0(s)(s-1)x^s + a_1(s+1)(s)x^{s+1} + \sum_{j=2}^{\infty} a_j(s+j)(s+j-1)x^{s+j}$$
$$+ a_0 sx^s + a_1(s+1)x^{s+1} + \sum_{j=2}^{\infty} a_j(s+j)x^{s+j}$$
$$+ \sum_{j=2}^{\infty} a_{j-2}x^{s+j} - a_0 n^2 s^s - a_1 n^2 x^{s+1} - \sum_{j=2}^{\infty} a_j n^2 x^{s+j} = 0$$

We can rewrite this equation as,

$$a_0(s(s-1)+s-n^2)x^s + a_1(s(s+1)+s+1-n^2)x^{s+1}$$
$$+ \sum_{j=2}^{\infty} a_j(s+j)(s+j-1)x^{s+j} + \sum_{j=2}^{\infty} a_j(s+j)x^{s+j} + \sum_{j=2}^{\infty} a_{j-2}x^{s+j} - \sum_{j=2}^{\infty} a_j n^2 x^{s+j} = 0$$

The coefficients of x^s is given by the following,

$$\begin{aligned} a_0(s(s-1)+s-n^2) &= a_0(s^2-s+s-n^2) \\ &= a_0(s^2-n^2) \end{aligned}$$

Here we obtain, $s^2 - n^2 = 0$ with solutions $s = \pm n$. This is our indicial equation and again $a_0 \neq 0$ by definition.

For the coefficients of the x^{s+1}, we can simplify it as,

$$\begin{aligned} a_1(s(s+1)+s+1-n^2) &= a_1(s^2+2s+1-n^2) \\ &= a_1((s+1)^2-n^2) \\ &= a_1((s+1)+n)((s+1)-n) \\ 0 &= a_1(s+1+n)(s+1-n) \end{aligned}$$

where we obtain the solutions $s = \{-(1+n), -(1-n)\}$. But for $s = \pm n$, neither of this equation vanishes, and thus we must require $a_1 = 0$.

The recurrence equation can be obtained from the last terms that have summations,

$$\begin{aligned} \sum_{j=2}^{\infty} a_j(s+j)(s+j-1)x^{s+j} + \sum_{j=2}^{\infty} a_j(s+j)x^{s+j} + \sum_{j=2}^{\infty} a_{j-2}x^{s+j} - \sum_{j=2}^{\infty} a_j n^2 x^{s+j} &= 0 \\ \sum_{j=2}^{\infty} \left(a_j(s+j)(s+j-1) + a_j(s+j) + a_{j-2} - a_j n^2 \right) x^{s+j} &= 0 \end{aligned}$$

For `s=n` we have the following recurrence equation:

$$\begin{aligned} a_j(n+j)(n+j-1) + a_j(n+j) + a_{j-2} - a_j n^2 &= 0 \\ a_j\left[(n+j)(n+j-1) + (n+j) - n^2\right] + a_{j-2} &= 0 \\ a_j\left[(n+j)^2 - (n+j) + (n+j) - n^2\right] + a_{j-2} &= 0 \\ a_j\left[(n+j)^2 - n^2\right] + a_{j-2} &= 0 \end{aligned}$$

We can adjust the index to start at zero by replacing j with $j+2$.

$$
\begin{aligned}
a_{j+2}\left[a_j\left[(n+j+2)^2-n^2\right]+a_j\right.&=0\\
a_{j+2}(n+j+2+n)(n+j+2-n)+a_j&=0\\
a_{j+2}(2n+j+2)(j+2)+a_j&=0\\
a_{j+2}&=-a_j\frac{1}{(2n+j+2)(j+2)}
\end{aligned}
$$

which is our desired recurrence relation. Repeated application of this recurrence relation leads to,

$$
\begin{aligned}
a_2&=-a_0\frac{1}{(2n+2)2}=-a_0\frac{1}{(n+1)2^2}\\
a_4&=-a_2\frac{1}{(2n+4)4}=a_0\frac{1}{(n+1)2^2}\frac{1}{2(n+2)2^2}=\frac{a_0}{2!(n+1)(n+2)2^4}\\
a_6&=-a_4\frac{1}{(2n+6)6}=-\frac{a_0n!}{2!(n+2)!2^4}\frac{1}{2^2(n+3)3}=\frac{a_0}{2^6 3!(n+1)(n+2)(n+3)}\\
&\vdots
\end{aligned}
$$

In general we can express the coefficients as,

$$
a_{2p}=(-1)^p\frac{a_0n!}{2^{2p}p!(n+p)!}
$$

For the odd-numbered coefficients,

$$
a_1=a_3=a_5=\cdots=0
$$

Inserting these coefficients in our series solution, we have,

$$
\begin{aligned}
y(x)&=x^s(a_0+a_1x+a_2x^2+a_3x^3+\cdots)\\
&=\sum_{j=0}a_jx^{s+j},a_0\neq 0 \qquad\qquad @s=n\\
&=x^na_0\left(1+0-\frac{n!x^2}{2^2 1!(n+1)!}+0+\frac{n!x^4}{2^4 2!(n+2)!}+\cdots\right)
\end{aligned}
$$

In summation form,

$$
\begin{aligned}
y(x)&=x^na_0\left(1-\frac{n!x^2}{2^2 1!(n+1)!}+\frac{n!x^4}{2^4 2!(n+2)!}+\cdots\right)\\
&=a_0x^n\left((-1)^0\frac{n!x^0}{2^0 0!(n+0)!}-\frac{n!x^2}{2^2 1!(n+1)!}+\frac{n!x^4}{2^4 2!(n+2)!}+\cdots\right)\\
&=a_0\sum_{j=0}^{\infty}(-1)^j\frac{n!x^{n+2j}}{2^j j!(n+j)!}\\
&=a_0n!2^n\sum_{j=0}^{\infty}(-1)^j\frac{x^{n+2j}}{2^{n+2j}j!(n+j)!}\\
&=a_0n!2^n\sum_{j=0}^{\infty}(-1)^j\frac{1}{j!(n+j)!}\frac{x^{n+2j}}{2^{n+2j}}\\
&=\underbrace{a_0n!2^n\sum_{j=0}^{\infty}(-1)^j\frac{1}{j!(n+j)!}\left(\frac{x}{2}\right)^{n+2j}}_{\text{Bessel function}}
\end{aligned}
$$

We can identify the Bessel's function J_n if $a_0 = 1/2^n n!$.

$$J_n(x) = \sum_{j=0}^{\infty}(-1)^j \frac{1}{j!(n+j)!}\left(\frac{x}{2}\right)^{n+2j}$$

Note that this equation, $J_n(x)$, has either even or odd symmetry. By even or odd symmetry we mean that the Bessel function is an even function if n is an even integer, while it is an odd function if n is odd; that is,

$$J_{2n}(x) = J_{2n}(-x)$$
$$J_{2n+1}(x) = -J_{2n+}(-x)$$

For example, lets say we have `n=0` ,then $J_0(x) = J_0(-x)$ by checking the form of the Bessel's equation.

$$\begin{aligned}
J_0(x) &= \sum_{j=0}^{\infty}(-1)^j \frac{1}{j!(j)!}\left(\frac{x}{2}\right)^{2j} \\
J_0(-x) &= \sum_{j=0}^{\infty}(-1)^j \frac{1}{j!(j)!}\left(\frac{-x}{2}\right)^{2j} \\
&= \sum_{j=0}^{\infty}(-1)^j \frac{1}{j!(j)!}(-1)^{2j}\left(\frac{x}{2}\right)^{2j} \\
&= \sum_{j=0}^{\infty}(-1)^j \frac{1}{j!(j)!}\left(\frac{x}{2}\right)^{2j} \\
J_0(x) &= J_0(-x)
\end{aligned}$$

Similar line of reasoning works when `n` is an odd number.

When `s=-n` and **n is not an intege**r, we may generate a second distinct series to be labeled $J_{-n}(x)$.

When $-n$ is **negative integer**, trouble develops. The recurrence relation for the coefficients a_j,

$$a_{j+2} = -a_j \frac{1}{(2n+j+2)(j+2)}$$

is now replaced by $-2n$.

$$a_{j+2} = -a_j \frac{1}{(-2n+j+2)(j+2)}$$

The problem with this is that when $j+2 = 2n$ or $j = 2(n-1)$, the coefficient a_{j+2} blows up (it diverges).

Now, Frobenius' method does not produce a solution consistent with our assumption that the series starts with x^{-n} .

Summary

- assuming a infinite series as solution, we have obtained two solutions for the linear oscillator equation and on for Bessel's equation (two if n is not an integer).
- The problem is that this method of series solution will not always work!

Regular and Irregular Singularities

The success of the series substitution method depends two things:

1. on the roots of the indicial equation
2. on the degree of singularity of the coefficients in the differential equations.

To better understand we will examine some ODE and apply this naive series substitution approach.

Consider the following,

$$y'' - \frac{6}{x^2}y = 0$$

We can rewrite the equation above as,

$$\begin{aligned}
0 &= x^2y'' - 6y \\
0 &= \sum_{j=0}^{\infty} a_j(s+j)(s+j-1)x^{s+j} - 6\sum_{j=0}^{\infty} a_j x^{s+j} \\
0 &= a_0(s)(s-1) - 6a_0 + \sum_{j=1}^{\infty} a_j(s+j)(s+j-1)x^{s+j} - 6\sum_{j=1}^{\infty} a_j x^{s+j} \\
0 &= a_0(s^2 - s - 6) + \sum_{j=1}^{\infty} a_j\Big((s+j)(s+j-1)x^{s+j} - 6\Big)x^{s+j}
\end{aligned}$$

The indicial equation then is,

$$s^2 - s - 6 = (s-3)(s+2) = 0$$

giving $s = 3$ and $s = -2$. For the coefficients, $j > 0$, all coefficients vanishes - there is no recurrence relation. However, we are left with two perfectly good solutions,

$$\begin{aligned}
y(x) &= x^s(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots) \\
&= \sum_{j=0} a_j x^{s+j}, a_0 \neq 0 \qquad\qquad @s = 3, -2 \\
&= x^s a_0\Big(1 + 0\Big) \\
&= x^3 \text{ and } x^{-2}
\end{aligned}$$

These two are our good solutions.

Now, consider the following equation,

$$y'' - \frac{6}{x^3}y = 0$$

Using power series, we have,

$$
\begin{aligned}
0 &= x^2 y'' - 6y \\
0 &= \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j+1} - 6 \sum_{j=0}^{\infty} a_j x^{s+j} \\
0 &= -6 a_0 x^s + \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j+1} - 6 \sum_{j=1}^{\infty} a_j x^{s+j}
\end{aligned}
$$

This sends the indicial equation to

$$-6a_0 = 0$$

with no solution at all since we've agreed that $a_0 \neq 0$.

Note that this ODE has an irregular singular point at the origin.

$$(x - x_0)^2 Q(x) = (x-0)^2 Q(x) = -x^2 \frac{6}{x^3} = -\frac{6}{x}$$

As $x \to 0$ the equation above diverges.

But not in the infinite range.

$$
\begin{aligned}
0 &= y'' - \frac{6}{x^3} y \\
0 &= z^4 w'' + 2z^3 w' + Q(z^{-1}) w \\
0 &= w'' + \frac{2}{z} w' - \frac{6}{z} w
\end{aligned}
$$

Lets add a term, y'/x to the first ODE and make it more general, that is we have,

$$y'' + \frac{y'}{x} - \frac{b^2}{x^2} y = 0$$

Using the power series, we have

$$
\begin{aligned}
0 &= y'' + \frac{y'}{x} - \frac{b^2}{x^2} y \\
0 &= x^2 y'' + x y' - b^2 y \\
0 &= x^2 \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} + x \sum_{j=0}^{\infty} a_j (s+j) x^{s+j-1} - b^2 \sum_{j=0}^{\infty} a_j x^{s+j} \\
0 &= \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j} + \sum_{j=0}^{\infty} a_j (s+j) x^{s+j} - b^2 \sum_{j=0}^{\infty} a_j x^{s+j} \\
0 &= a_0 \{s(s-1) + s - b^2\} x^s + \sum_{j=1}^{\infty} a_j (s+j)(s+j-1) x^{s+j} + \sum_{j=1}^{\infty} a_j (s+j) x^{s+j} - b^2 \sum_{j=1}^{\infty} a_j x^{s+j} \\
0 &= a_0 \{s^2 - b^2\} x^s + \sum_{j=1}^{\infty} a_j (s+j)(s+j-1) x^{s+j} + \sum_{j=1}^{\infty} a_j (s+j) x^{s+j} - b^2 \sum_{j=1}^{\infty} a_j x^{s+j}
\end{aligned}
$$

The indicial equation is

$$s^2 - b^2 = 0$$

and again, there is no recurrence relation. Solutions of the indicial equation above is $s = \pm b$. So we have the following solution for our ODE.

$$
\begin{aligned}
y(x) &= x^s (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \\
&= \sum_{j=0} a_j x^{s+j}, a_0 \neq 0 && @s = b, -b \\
&= x^s a_0 \Big(1 + 0\Big) \\
&= x^b \text{ and } x^{-b}
\end{aligned}
$$

If we change the power of x in the coefficient of y' from -1 to -2 there is a drastic change in the solution.

$$
y'' + \frac{y'}{x^2} - \frac{b^2}{x^2} y = 0
$$

Our indicial equation becomes $s = 0$ as shown by what follows,

$$
\begin{aligned}
&0 = y'' + \frac{y'}{x^2} - \frac{b^2}{x^2} y \\
&0 = x^2 y'' + x y' - b^2 y \\
&0 = x^2 \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j-2} + \sum_{j=0}^{\infty} a_j (s+j) x^{s+j-1} - b^2 \sum_{j=0}^{\infty} a_j x^{s+j} \\
&0 = \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j} + \sum_{j=0}^{\infty} a_j (s+j) x^{s+j-1} - b^2 \sum_{j=0}^{\infty} a_j x^{s+j} \\
&0 = \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j} + \sum_{j+1=0}^{\infty} a_{j+1} (s+j+1) x^{s+j} - b^2 \sum_{j=0}^{\infty} a_j x^{s+j} \\
&0 = a_0 s x^{s-1} + \sum_{j=0}^{\infty} a_j (s+j)(s+j-1) x^{s+j} + \sum_{j=0}^{\infty} a_{j+1} (s+j+1) x^{s+j} - b^2 \sum_{j=0}^{\infty} a_j x^{s+j} \\
&0 = a_0 s x^{s-1} + \sum_{j=0}^{\infty} \Big\{ a_j (s+j)(s+j-1) + a_{j+1} (s+j+1) - b^2 a_j \Big\} x^{s+j}
\end{aligned}
$$

However there is a recurrence relation.

$$
\begin{aligned}
a_{j+1} &= a_j \frac{b^2 - (s+j)(s+j-1)}{(s+j+1)} \\
&= a_j \frac{b^2 - j(j-1)}{(j+1)} && @s = 0
\end{aligned}
$$

If b is some constant, we will see that the series diverges

$$
\begin{aligned}
\lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right| &= \lim_{j \to \infty} \left| \frac{j(j+1)}{j} \right| \\
&= \lim_{j \to \infty} \left| \frac{j^2 + j)}{j} \right| \\
&= \lim_{j \to \infty} \left| \frac{j^2}{j} \right| = \lim_{j \to \infty} \Big| j \Big| = \infty
\end{aligned}
$$

Hence our series solution diverges for all $x \neq 0$. Again our method worked for the 3rd ODE with a regular singularity but failed when we had the irregular singularity.

Fuchs Theorem

When can the method of series substitution expected to work?

This basic question is answered by Fuchs' theorem.

> It asserts that we can always obtain at least one power-series solution, provided that we are expanding about a point which is an ordinary point or at worst a regular singular point. If we attempt an expansion about an irregular or essential singularity, our method may fail.

The good thing is, most important equations of mathematical physics have no irregular singularities in the finite plane.

As a further illustration of Fuchs' theorem:

- Legendre's equation (with ∞ as a regular singularity) has a convergent series solution in negative powers of the argument
- Bessel's equation (with an irregular singularity at infinity) yields asymptotic series. Although only asymptotic, these solutions are nevertheless extremely useful.

Summary

- if we are expanding about an ordinary point or worst about a regular singularity, the series substitution approach will yield at least one solution (Fuchs' theorem)
- whether we get one or two distinct solutions depends on the roots of the indicial equation
  - if the **two roots of the indicial are equal**, we can obtain only one solution by this series substitution method
  - if the **two roots differ by a nonintegral number**, two independent solutions may be obtained
  - if the **two roots differ by an integer**, the larger of the two will yield a solution, while the smaller may or may not give a solution, depending on the behavior of the coefficients.
- The usefulness of series solutions for numerical work depends on the rapidity of convergence of the series and the availability of the coefficients.
- **For numerical work, a direct numerical integration will be preferred.**

Other Solutions

1. Linear Independence of Solutions
   . Number of Solutions
   . Finding a Second Solution
   . Series Form of the Second Solution

Inhomogeneous Linear ODEs

1. Variation of Parameters

Nonlinear Differential Equations

1. Bernoulli and Riccati Equations
   . Fixed and Movable Singularities, Special Solutions

Sturm-Liouville Theory

Partial Differential Equations

Complex Variable Theory Some of the most powerful and widely useful tools in all of analysis Why complex variables are important

1. 2D, the electric potential viewed as a solution of Laplace's equation can be written as the real part of a complex-valued function
   . Uses various features of complex variable theory (e.g conformal mapping)to obtain formal solutions to a wide variety of electrostatics problems
   . Time-independent Schrodinger equation of quantum mechanics contains the imaginary unit, and its solutions are complex
   . Solving second-order differential equations of interest in physics using power series in z
   . Transforming Helmholtz equation into the time-independent diffusion equation
   . Connecting spherical and hyperbolic trigonometric functions
   . Transforms Bessel functions into their modified counterparts
   . Other useful applications
   . Evaluating definite integrals and infinite series
   . Inverting power series
   . Forming infinite products
   . Obtaining solutions of D.E. for large values of the variable
   . Investing the stability of potentially oscillatory systems
   . Inverting integral transforms
   COMPLEX VARIABLES AND FUNCTIONS

Complex numbers $z = x + iy$ as ordered pairs of two real numbers, x and y.

- Complex conjugate
- Argand diagram
- Modulus
- Argument
- Multivalued functions

CAUCHY-RIEMANN CONDITIONS

- Derivative of $f(z)$
- Cauchy-Riemann conditions
- Analytic Functions
- A function $f(z)$ is differentiable and single-valued in a region of the complex plane
- Singular point