Chapter 7: Ordinary Differential Equations

7.5 Exercises

`7.5.1`

`7.5.7` A quantum mechanical analysis of the Stark effect (parabolic coordinates) leads to the differential equation

$$\frac{d}{d\xi}\left(\xi\frac{du}{d\xi}\right)+\left(\frac{1}{2}E\xi+\alpha-\frac{m^2}{4\xi}-\frac{1}{4}F\xi^2\right)u=0 \tag{1}$$

Here α is a constant, E is the total energy, and F is a constant such that Fz is the potential energy added to the system by the introduction of an electric field.

Using the larger root of the indicial equation, develop a power-series solution about $\xi = 0$. Evaluate the first three coefficients in terms of a_0.

Solution :

$$0=\frac{d}{d\xi}\left(\xi\frac{du}{d\xi}\right)+\left(\frac{1}{2}E\xi+\alpha-\frac{m^2}{4\xi}-\frac{1}{4}F\xi^2\right)u \tag{2}$$

$$0=\xi\frac{d^2u}{d\xi^2}+\frac{du}{d\xi}+\left(\frac{1}{2}E\xi+\alpha-\frac{m^2}{4\xi}-\frac{1}{4}F\xi^2\right)u \tag{3}$$

applying our series solution of the form: $\sum_{j=0}^{\infty} a_j\xi^{s+j}$. We have,

$$0=\xi\frac{d^2u}{d\xi^2}+\frac{du}{d\xi}+\left(\frac{1}{2}E\xi+\alpha-\frac{m^2}{4\xi}-\frac{1}{4}F\xi^2\right)u \tag{4}$$

$$0=\xi\sum_{j=0}^{\infty}a_j(s+j)(s+j-1)\xi^{s+j-2}+\sum_{j=0}^{\infty}a_j(s+j)\xi^{s+j-1}+\left(\frac{1}{2}E\xi+\alpha-\frac{m^2}{4\xi}-\frac{1}{4}F\xi^2\right)\sum_{j=0}^{\infty}a_j\xi^{s+j} \tag{5}$$

we can simplify this as follows,

$$0 = \xi \sum_{j=0}^{\infty} a_j (s+j)(s+j-1)\xi^{s+j-2} + \sum_{j=0}^{\infty} a_j (s+j)\xi^{s+j-1} + \left(\frac{1}{2}E\xi + \alpha - \frac{m^2}{4\xi} - \frac{1}{4}F\xi^2\right) \sum_{j=0}^{\infty} a_j \xi^{s+j} \quad (6)$$

$$0 = \sum_{j=0}^{\infty} a_j (s+j)(s+j-1)\xi^{s+j-1} + \sum_{j=0}^{\infty} a_j (s+j)\xi^{s+j-1} + \frac{1}{2}E \sum_{j=0}^{\infty} a_j \xi^{s+j+1} + \alpha \sum_{j=0}^{\infty} a_j \xi^{s+j} \quad (7)$$

$$- \frac{m^2}{4} \sum_{j=0}^{\infty} a_j \xi^{s+j-1} - \frac{F}{4} \sum_{j=0}^{\infty} a_j \xi^{s+j+2} \quad (8)$$

$$0 = \sum_{j=0}^{\infty} a_j \left[(s+j)(s+j-1) + (s+j) - \frac{m^2}{4}\right] \xi^{s+j-1} + \frac{1}{2}E \sum_{j=0}^{\infty} a_j \xi^{s+j+1} \quad (9)$$

$$+ \alpha \sum_{j=0}^{\infty} a_j \xi^{s+j} - \frac{F}{4} \sum_{j=0}^{\infty} a_j \xi^{s+j+2} \quad (10)$$

$$0 = a_0 \{s(s-1) + s - \frac{m^2}{4}\} \xi^{s-1} + \sum_{j=1}^{\infty} a_j \left[(s+j)(s+j-1) + (s+j) - \frac{m^2}{4}\right] \xi^{s+j-1} \quad (11)$$

$$+ \frac{1}{2}E \sum_{j=0}^{\infty} a_j \xi^{s+j+1} + \alpha \sum_{j=0}^{\infty} a_j \xi^{s+j} - \frac{F}{4} \sum_{j=0}^{\infty} a_j \xi^{s+j+2} \quad (12)$$

$$0 = a_0 \{s(s-1) + s - \frac{m^2}{4}\} \xi^{s-1} + \sum_{j=0}^{\infty} a_{j+1} \left[(s+j+1)(s+j) + (s+j+1) - \frac{m^2}{4}\right] \xi^{s+j} \quad (13)$$

$$+ \frac{1}{2}E \sum_{j=1}^{\infty} a_{j-1} \xi^{s+j} + \alpha \sum_{j=0}^{\infty} a_j \xi^{s+j} - \frac{F}{4} \sum_{j=2}^{\infty} a_{j+2} \xi^{s+j} \quad (14)$$

Our indicial equation is given as,

$$s^2 - \frac{m^2}{4} = 0 \quad (15)$$

the solutions are $s = \pm\frac{m}{2}$

The remaining terms can be simplified as follows,

$$0 = a_0 \{s(s-1) + s - \frac{m^2}{4}\} + \sum_{j=0}^{\infty} a_{j+1} \left[(s+j+1)(s+j) + (s+j+1) - \frac{m^2}{4}\right] \xi^{s+j} + \frac{1}{2}E \sum_{j=1}^{\infty} a_{j-1} \xi^{s+j} \quad (16)$$

$$+ \alpha \sum_{j=0}^{\infty} a_j \xi^{s+j} - \frac{F}{4} \sum_{j=2}^{\infty} a_{j+2} \xi^{s+j} \quad (17)$$

$$0 = a_0 \{s(s-1) + s - \frac{m^2}{4}\} \xi^{s-1} + a_1 [(s+1)s + (s+1) - \frac{m^2}{4}] \xi^s + a_2 [(s+2)(s+1) \quad (18)$$

$$+ (s+2) - \frac{m^2}{4}] s^{s+1} + a_0 \frac{E}{2} \xi^{s+1} + \alpha a_0 \xi^s + \alpha a_1 \xi^{s+1} + \cdots \quad (19)$$

And combining like power coefficients, first for ξ^s we have,

$$0 = a_1 [(s+1)s + (s+1) - \frac{m^2}{4}] + \alpha a_0 \qquad @s = m/2 \quad (20)$$

$$0 = a_1 [(s^2 + 2s + 1 - \frac{m^2}{4}] + \alpha a_0 \quad (21)$$

$$0 = a_1 [(m/2)^2 + 2(m/2) + 1 - \frac{m^2}{4}] + \alpha a_0 \quad (22)$$

$$0 = a_1 [m+1] + \alpha a_0 \quad (23)$$

So our first coefficient $a_1 = -a_0 \frac{\alpha}{m+1}$. For the coefficient of ξ^{s+1} we have,

$$0 = a_2[s+2)(s+1) + (s+2) - \frac{m^2}{4}] + a_0\frac{E}{2} + \alpha a_1 \tag{24}$$

$$0 = a_2[s^2 + 4s + 4 - \frac{m^2}{4}] + a_0\frac{E}{2} + \alpha a_1 \qquad @s = m/2 \tag{25}$$

$$0 = a_2 2[m+2] + a_0\frac{E}{2} - a_0\alpha\frac{\alpha}{m+1} \tag{26}$$

Hence for our third coefficient we have,

$$a_2 = a_0\left[\frac{\alpha}{2(m+2)(m+1)} - \frac{E}{4(m+2)}\right] \tag{27}$$

Our solution with the first three coefficients in terms of a_0 is given:

$$y(x) = x^s(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad @s = m/2, x \to \xi \tag{28}$$

$$y(x) = x^{m/2}(a_0 + \left[-a_0\frac{\alpha}{m+1}\right]\xi + a_0\left[\frac{\alpha}{2(m+2)(m+1)} - \frac{E}{4(m+2)}\right]\xi^2 + \cdots) \tag{29}$$

$$y(x) = a_0 x^{m/2}\left\{1 - \frac{\alpha}{m+1}\xi + \left[\frac{\alpha}{2(m+2)(m+1)} - \frac{E}{4(m+2)}\right]\xi^2 + \cdots)\right\} \tag{30}$$

Note that the perturbation F does not appear until a_3 is included.

`7.5.8` For the special case of no azimuthal dependence, the quantum mechanical analysis of the hydrogen molecular ion leads to the equation

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{du}{d\eta}\right] + \alpha u + \beta\eta^2 u = 0 \tag{31}$$

Develop a power-series solution for $u(\eta)$. Evaluate the first three nonvanishing coefficients in terms of a_0.

ANS: indicial equation $s(s-1) = 0$

Solution:

$$u_{k=1} = a_0\eta\left\{1 + \frac{2-\alpha}{6}\eta^2 + \left[\frac{(2-\alpha)(12-\alpha)}{120} - \frac{\beta}{20}\right]\eta^4 + \cdots\right\} \tag{32}$$